9.14 Sannsynsmaksimeringsprinsippet

Eks. Slår mynt og krone 10 ganger

La X vere talet på mynt. $X \sim B(10, p)$. Vil estimere p. Observerer 6 mynt: $x = 6$

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$P(X=6)$	0	0.005	0.036	0.111	0.205	0.251	0.200	0.088	0.011

Litt rekning.

Observerer $X = x$. Vil finne den p som maksimerer

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x} = c p^x (1-p)^{n-x}$$

$$\frac{d}{dp} P(X=x) = 0 \iff c x p^{x-1}(1-p)^{n-x} - c(n-x)(1-p)^{n-x-1} p^x = 0$$

$$\Rightarrow c p^{x-1}(1-p)^{n-x-1}\left(x(1-p) - (n-x)p\right) = 0$$

$$\Rightarrow x - px - np + px = 0 \Rightarrow \hat{p}_e = \frac{x}{n}$$ er det estimatet for p som maksimerer $P(X=x)$

$\hat{p} = \frac{X}{n}$ blir kalla sannsynsmaksimeringsestimatoren for p | NB! $\frac{d}{dp} P(X=x) \geq 0$ for $p < \frac{x}{n}$ og < 0 for $p > \frac{x}{n}$

Generelt. La $X_1, X_2, \ldots, X_n$ vere uavhengige med $\left\{\begin{matrix}\text{punktsannsyn} \\ \text{sannsynstettleik}\end{matrix}\right\}$ $f(x;\theta)$. Simultan sannsynsfordeling er då gitt ved: $f(x_1, x_2, \ldots, x_n; \underline{\theta}) = f(x_1;\underline{\theta}) \cdot f(x_2;\underline{\theta}) \cdot \ldots \cdot f(x_n;\underline{\theta})$

Definisjon

La $X_1, X_2, \dots, X_n$ vere uavh. stokastiske variable. Funksjonen

$$L(x_1, x_2, \dots, x_n; \underline{\theta}) = \begin{cases} P(X_1 = x_1) \cdot P(X_2 = x_2) \cdot \dots \cdot P(X_n = x_n), & \text{diskret} \\ f(x_1; \underline{\theta}) \cdot f(x_2; \underline{\theta}) \cdot \dots \cdot f(x_n; \underline{\theta}), & \text{kontinuerleg} \end{cases}$$

som funksjon av $\underline{\theta}$ gitt dataane blir kalla rimeleg-funksjonen. Sannsynsmaksimeringsestimatet for $\underline{\theta}$, $\hat{\underline{\theta}}(x_1, \dots, x_n)$ finn ein ved å maksimere denne m.o.p. $\underline{\theta}$

Sannsynsmaksimeringsestimatoren er gitt ved $\hat{\theta}(X_1, X_2, \dots, X_n)$

Eks. Poissonfordeling, $X_1, \dots, X_n$ tilf. utval

$$f(x; \mu) = \frac{\mu^x e^{-\mu}}{x!}, \quad x = 0, 1, 2, \dots \qquad \mu = \lambda t$$

$$L(x_1, x_2, \dots, x_n; \mu) = \prod_{i=1}^{n} \frac{\mu^{x_i} e^{-\mu}}{x_i!} = \frac{e^{-n\mu} \mu^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\ln L = -n\mu + \sum_{i=1}^{n} x_i \ln(\mu) - \ln\left(\prod_{i=1}^{n} x_i!\right)$$

$$\frac{d \ln L}{d\mu} = -n + \sum_{i=1}^{n} \frac{x_i}{\mu} = 0 \Leftrightarrow \mu = \frac{\sum_{i=1}^{n} x_i}{n} \quad \text{ɔ: sannsynsmaksimerings-estimatet for}$$

μ er gitt ved $\hat{\mu}_n = \bar{x}$

stok. variable.

Sannsynsmaksimeringsestimatoren blir $\hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}$

$$\frac{d^2 \ln L}{d\mu^2} = \frac{-\sum_{i=1}^{n} x_i}{\mu^2} < 0 \Rightarrow \text{maksimum}$$

Eks.

La $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2, \dots, n$ og uavh.

ɔ: $f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $\quad -\infty < x < \infty$

$$L(x_1, x_2, \dots, x_n; \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot \frac{1}{\sigma^n} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$\ln L(x_1, x_2, \dots, x_n; \mu, \sigma^2) = -n \ln\sqrt{2\pi} - \frac{n}{2}\ln\sigma^2 - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

$$\frac{\partial \ln L}{\partial \mu} = \frac{1}{2} \cdot \frac{2}{\sigma^2} \sum_{i=1}^{n}(x_i - \mu) = 0 \iff \sum_{i=1}^{n} x_i - n\mu = 0 \qquad (1)$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2}\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^4} = 0 \iff -n\sigma^2 + \sum_{i=1}^{n}(x_i-\mu)^2 = 0 \qquad (2)$$

Frå (1) $\hat{\mu}_e = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$ som innsett i (2) gjev

$$\hat{\sigma}_e^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

Estimatorane blir: $\hat{\mu} = \bar{X}$, $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$

NB! $\hat{\sigma}^2$ er ikkje forventningsrett.

La X_i, $i = 1, 2, \dots, n$ vere uavh og ha sannsynstettleik

$$f(x; \theta) = \begin{cases} \frac{\theta}{x^{\theta+1}}, & x > 1 \\ 0, & \text{elles.} \end{cases}$$

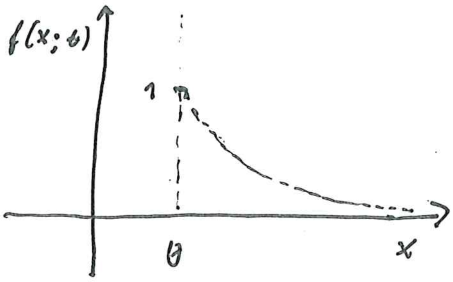
$$L(x_1, x_2, \dots, x_n; \theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^{\theta+1}} = \frac{\theta^n}{\left(\prod_{i=1}^{n} x_i\right)^{\theta+1}}$$

$$\Rightarrow \ln L(x_1, x_2, \dots, x_n; \theta) = n \ln\theta - (\theta+1)\sum_{i=1}^{n} \ln x_i$$

$$\frac{d \ln L}{d\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \hat{\theta}_e = \frac{n}{\sum_{i=1}^{n} \ln x_i}, \quad \hat{\theta} = \frac{n}{\sum_{i=1}^{n} \ln X_i}$$

La $X_1, \ldots, X_n$ vere uavhengige med sannsynstettleik

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, & x > \theta \\ 0, & \text{elles} \end{cases}$$



$$L(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)}$$

$$= e^{-\sum_{i=1}^{n} x_i + n\theta}$$

$$\ln L(x_1, x_2, \ldots, x_n; \theta) = n\theta - \sum_{i=1}^{n} x_i$$

$$\frac{d \ln L}{d\theta} = n$$

$\Rightarrow \hat{\theta}_e = \min(x_1, \ldots, x_n)$ og $\hat{\theta} = \min(X_1, \ldots, X_n)$